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SOME ALMOST CONNECTED LOCALLY COMPACT GROUPS WHOSE GROUP TOPOLOGY IS DETERMINED BY THE BOUNDED STRUCTURE

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Abstract. It is proved that the topology of an almost connected locally compact group whose quotient group by the identity component is finitely generated and the identity component is a perfect connected locally compact group is determined by the bounded structure in the class of locally compact group topologies.

§ 1. Introduction

Continuing our investigations concerning groups whose topology is unique in some class [1–3], we discuss here conditions under which the topology of an almost connected locally compact group is determined by the bounded structure in the class of locally compact group topologies. There are important classes of almost connected locally compact groups whose topology is not unique in the class under consideration [4, 5], and therefore the problem is of interest.

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§ 2. Preliminaries

Recall two important facts.

Lemma 1 ([2], Corollary)**.** *If a locally compact group G is connected and perfect and if π is a locally bounded and finally continuous homomorphism of G into a connected locally compact group H, then π is continuous.*

Lemma 2 ([6], see also [7])**.** *If H is a subgroup of finite index in a finitely generated profinite group G, then H is necessarily an open subgroup of G.*

In particular, this means that the family of open subgroups of G coincides with the family of subgroups of finite index.

Lemma 3. *Let G be a finitely generated profinite group G. The group topology of G is defined uniquely.*

Proof. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are profinite group topologies on G, then every subgroup of finite index is closed with respect to both $\mathcal{T}_1$ and $\mathcal{T}_2$. Hence, by Lemma 2, the family of subgroups of finite index forms a base of open neighborhoods of the identity element in both the topologies. Therefore, the topologies coincide.

§ 3. Main theorem

Theorem. *Let G be an almost connected locally compact group whose totally disconnected quotient group by the identity component G_0 of G is finitely generated and the identity component G_0 is a perfect connected locally compact group. Then every locally bounded* (*not necessarily continuous a priori*) *isomorphism G onto itself is automatically continuous.*

Proof. Let the assumptions of the theorem be satisfied for some almost connected locally compact group G. Every closed subgroup of finite index obviously contains the identity component G_0 of G, and the intersection of these subgroups coincides with the identity component G_0. Hence, by Lemma 3, the topology of the compact totally disconnected quotient group G/G_0 is defined uniquely, and the identity component G_0 does not depend on the choice of a group topology satisfying the conditions of the theorem. Therefore, if π is a locally bounded isomorphism of G onto itself, then G_0 is subjected to a locally bounded isomorphism onto itself. By the assumption of the theorem and by Lemma 1, the restriction of the isomorphism to G_0 is continuous, and thus a homeomorphism of G_0 onto itself.

Assume that G is a Lie group with finite quotient group G/G_0. Let $\{g_\alpha\}$, $\alpha \in A$, be a net in G convergent to $g_0 \in G$. Then $\rho(g_\alpha)$ converges to $\rho(g_0)$ in G/G_0, where ρ stands for the canonical quotient mapping $\rho\colon G \to G/G_0$. By the corresponding Lee theorem (Theorem 2.12 of [8]), every Lie group with finite quotient group admits a finite group D for which $G = G_0 D$. Let $g_\alpha = g_{0,\alpha} d_\alpha$ for some $g_{0,\alpha}$ in G_0 and d_α in D. Since D is compact, we may choose a convergent subnet of d_α, which we denote by the same symbol. Then $\{g_\alpha\}$ converges to g_0 in G and $\{d_\alpha\}$ converges to some d_0 in D, and hence $g_{0,\alpha}$ converges to some g_{00} in G. However, G_0 is closed, $g_{0,\alpha} \in G_0$, and therefore $g_{00} \in G_0$ and $g_0 = g_{00} d_0$. Since $D \cup G_0$ is finite, it follows that the convergence in D is completely characterized by the convergence in the quotient group G/G_0, and thus the assertion of the theorem holds for all Lie groups with finite quotient groups by the perfect identity components.

Let us return to the general case. The compact subgroups of G_0 are defined uniquely by the bounded structure. Consider the family of compact normal subgroups N of G_0 for which the quotient G_0/N is a Lie group. Since G/G_0 is compact, it follows that the subgroups N may be assumed to be invariant with respect to the inner automorphisms of G. The connected components of the identity elements $\{G/N\}_0$ of the quotient groups G/N are Lie groups, and therefore, by another Lee's theorem (Theorem 2.13 of [8]), the group G/N can be represented as the product $\{G/N\}_0 D$ for some totally disconnected group D. The intersection $D \cap \{G/N\}_0$ is a totally disconnected subgroup of the Lie group $\{G/N\}_0 D$. Hence, this group is finite. Since the topology of the totally disconnected compact quotient group $D/\{G/N\}_0 \cap D$ is unique by Lemma 3 and the topology of $\{G/N\}_0$ is determined by the bounded structure, it follows that the "abstract" isomorphism π of G onto itself is automatically continuous.

Remark 1. It follows immediately from the proof of the theorem that, for every group G satisfying the conditions of the theorem, an "abstract" isomorphism of G onto itself is continuous if the restriction of the isomorphism to the connected component of G is locally bounded.

Corollary. *The topology of an almost connected locally compact group G whose quotient group by the identity component is finitely generated and the identity component is a perfect connected locally compact group is determined in the class of locally compact group topologies by the structure of bounded sets of the connected component of G.*

Proof. This follows immediately from the theorem.

§ 4. Comment

Remark 2. The conditions of the theorem are substantial.

Proof. The condition that the compact quotient group is finitely generated is substantial because, if it is violated, then the topology of the quotient group can be not unique (Lemma 2.3 of [4]; see also [9]). The condition that G_0 is perfect is substantial by the corollary in [2].

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